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FREGE'S LOGIC

IN VIEW of the recognition generally accorded Frege as a pioneer in the field of symbolic or mathematical logic, it is a surprising fact that no systematic survey of his work as a whole has as yet been undertaken. Perhaps the nearest approach to such a study is a very interesting and full exposition and discussion of certain of Frege's basic doctrines in an Appendix to Bertrand Russell's *Principles of Mathematics;* but this, like the other discussions extant, does not pretend to present a clear general account of just what Frege was trying to do, or a critical estimate of how near he came to accomplishing it. The following article represents an attempt partly to fill this gap in the recent history of logic.

The principal works of Gottlob Frege (1848-1925) include the *Begriffschrift, eine arithmetische nachgebildete Formalsprache des reinen Denkens* (1879; later partly repudiated), *Die Grundlagen der Arithmetik* (1884, reprinted 1934), *Funktion und Begriff* (1891), and the *Grundgesetze der Arithmetik* (2 vols. 1893-1903), together with supplementary articles in various periodicals. An English translation of a small portion of the last mentioned work, by Johann Stachelroth and P. E. B. Jourdain, appeared in *The Monist* (Vols. XXV (1925), XXVI (1926), and XXVII (1927)).

According to Frege's great admirer, Bertrand Russell, the *Grundlagen* marks the beginning of "the logical theory of arithmetic". The first step, therefore, towards acquiring an understanding of the import of Frege's work for logic and mathematics must be to determine as precisely as possible the meaning of this phrase. As the title of the book indicates clearly enough, its author is explicitly concerned with the logic of mathematics, just as other scientists, before and since, have been concerned, among other things, with the logic of physics or biology, as the case may be. But in Frege's case this concern led in a certain direction, and issued in results of considerable significance for the future of logic, and more especially of symbolic logic, itself, as well as for the special science in question.

The starting-point of his studies was his interest in the demand which had steadily been growing stronger for some time, for more rigorous proofs of mathematical propositions, and a more precise determination of the limits of their validity. The obvious need for accurate definitions in this context had led to new conceptions of functions, of infinity, and of negative and irrational numbers. And it led finally, in the work of Cantor, Dedekind, and Frege himself, to a radical attempt to define even the simple whole numbers themselves, and to prove the simplest laws binding these numbers together in a system.

And with Frege at any rate, the urge to go further yet, to search out and wherever possible to come to some definitive conclusions about the philosophical and methodological presuppositions of his science was insistent and dominant over all other motives.

Taking first the problem of definition, Frege finds that previous attempts at a definition of number have presupposed erroneous theories concerning what kind of object it is to which number can properly be ascribed, and that the resultant definitions have naturally enough failed of their purpose. Upon examination it will be found, according to this authority, that these theories fall into two groups, namely that number it to be ascribed to physical objects (Cantor, Schroeder, and especially Mill), or to psychological ones (Berkeley, Dedekind, *et al.*).

But physical entities may be regarded either as one or as many—*two* boots or *one* pair, thousands of leaves or *the* foliage of a tree, etc. Therefore Frege argues that it is impossible to predicate unambiguous numbers of such objects, or to define the numbers themselves unequivocally. Neither is the view that number is an object of psychology, or the resultant of psychical processes, at all tenable. Whoever determines the area of the North Sea as so many square miles, or the number of petals of a rose, is plainly not referring to a psychical state or process, but is stating a fact as objective as the saltiness of the sea or the color of the rose. What in fact is common to number and saltiness, or to number and color, for example, is not that they are both sensibly perceptible in external things, but that they are both objective. In fine, number is neither spatial and physical, nor subjective and mental, but non-sensible and objective, like the earth's axis or the center of the solar system (*Grundlagen,* sec. 26). Thinking, it seems plain to Frege, creates none of these 'objectives'; rather, they are eternally 'there', metaphorically speaking, to be thought about. Hence any attempt either to determine the nature of number genetically, or to trace the historical development of the number-concept, and in this wise to inquire into its possible derivation from more elementary but less precise ideas, Frege rules out *ab initio* as beside the mark. Like Bolzano's ideas-in-themselves, which Frege's 'objectives' quite closely resemble, mathematical entities are thus assigned to a special, sacrosanct realm of being, not subject to the vicissitudes of this earth, which the fortunate spectator may discover and contemplate, and, if constituted like Bertrand Russell's "free man", worship from afar.

So far, then, the position has been reached that number is a predicate, neither of physical things nor of psychological ideas, but of objective concepts. In 'The earth has one satellite', 'one' may properly be ascribed, not to the moon itself (which may just as well be regarded as 'many' molecules), but to the concept or general term 'earth's satellite'. Still more clearly, in 'Venus has 0 satellites', '0' is a property, not of a physical object (for to what physical object could the number 0 apply?) but of the concept 'satellite of Venus'.

Here, incidentally, Frege points out that it is important to distinguish between the proper name of a thing, *e.g.*, 'Venus', and words signifying concepts; and especially is this so where only one object "falls under" the concept. Obviously it is nonsense to ask what objects fall under a proper name. 'The moon' is the proper name of the object falling under the concept designated by the words 'satellite of the earth' (*Grundlagen,* secs. 45-54).

Now it seems to be Frege's contention, though this is nowhere expressed with anything like the desirable clarity or convincing argumentation, that concepts, 'objectives', being neither mental nor physical, can therefore only be described as purely logical entities. It follows at once from such considerations, that number, as predicable of, or applicable to concepts, is also to be included within the general sphere of logic. Or, to put it in another way, since the subject matter of both logic and mathematics belongs to the same realm, they are to all intents and purposes inseparable, if not identical sciences.

Since at all events number is applicable to concepts, and since concepts possess objective existence, in the sense elucidated above, existence and number may be said to have a certain something in common. In fact, "the affirmation of existence (presumably in the logical and mathematical sense of the word) is nothing other than the denial of the number zero (*Nullzahl*)" (*Grundlagen,* sec. 53)! Further consideration of the way in which numbers occur in statements shows, so Frege contends, that the copula 'is' of such statements is really an abbreviation for 'is equal to', or 'is the same as'. 'Jupiter has four moons' means 'The number of moons of Jupiter is four'; and this means, in turn, that the phrase 'the number of Jupiter's moons' designates the same 'object' as the word 'four' (sec. 57). Numbers, then, Frege concludes, designate "independent objects" (*selbstaendige Gegenstaende*).

Now it happens to be the case, however, that one genus of propositions is that which gives expression to a re-cognition (*Wiedererkennen*) of something proviously cognized. Hence if 'a' is to designate an 'independent object' unambiguously, there must be a criterion which will enable one to decide whether 'b' designates the same object or not. That is to say, in order to define numbers "we must clarify the meaning of the proposition, 'the number (*Zahl*) which applies to the concept F is the same as that which applies to the concept C'; *i.e.,* we must reproduce the content of this proposition in another manner, without using the expression, 'the number (*Anzahl*) which applies to the concept F'" (sec. 62). According to Frege it will be found that this clarification will further yield a general criterion for determining the equality of numbers, for grasping a determinate number as such, and for bestowing a proper name upon it.

But to be able to assert the proposition just formulated is tantamoun to being able to answer the question: When do the concepts (*e.g.,* F and C above) applicable to two collections of objects (*e.g.,* those 'falling under' concepts F and C) have the same number of terms—or, as would ordinarily be said, the same extension? And the answer is: When there obtains a one-to-one relation between all the terms of the one collection and all the terms of the other, taken severally (such as that which holds, for example, of the collections 'husbands' and 'wives' in monogamous countries). Finally this "similarity" of two such collections—to use Russellian language—leads to the definition of the number of a given collection (extension of a given concept) as the class of all collections that are similar (stand in a one-one relation) to it; or, more precisely, 'the number of terms in a given class' is defined as the equivalent of 'the class of all classes that are similar to a given class'. Thus 'two' is 'the class of all couples'; 'three', 'the class of all triads', and so forth. Frege contends that this extensional definition follows from, and confirms, his view that numbers are to be predicated of concepts, that it yields the usual arithmetical properties of numbers, finite and infinite alike, and that it applies to '0' and '1', which are often treated as special cases, in the same manner as to all other numbers (cf. *Grundlagen*, sec. 62 ff.).

A rather obvious criticism of this definition, namely that it is circular, in that it presupposes the very terms it professes to define—although this circularity may be concealed by an elaborate technical circumlocution—has been advanced by various students of the logic of mathematics (*e.g.*, Kerry, Cassirer, Poincaré); but Frege and Russell have both rather abruptly brushed this criticism aside as based on a misunderstanding, and there, to all intents and purposes, the matter rests at the present time. Nevertheless, a feeling that all of the issues here in question have not yet been met as squarely as could be desired is bound to linger in the minds of students both of logic and of mathematics. And from many authorities—Husserl, Fraenkel, Brunschvicg, Spaier, Meyerson, to mention only a few—have come criticisms directed against various aspects of this whole extremely abstruse line of thought.

Why, for example, should it be *assumed*, without argument, that number must be a 'property'—to use the Frege-Russell terminology—or intrinsically 'applicable to', 'predicable of' anything at all? In fact do not the very considerations adduced by

Frege himself all but positively declare that number is essentially an extrinsic or relational determination of the phenomenal objects of experience, and not 'predicable of' them either ambiguously or unambiguously, for that very reason? The leaves of a tree are possessed of various properties *qua* leaves, or instrinsically; but they are not numerable *qua* leaves, but rather only because and in so far as they are phenomenal objects qualitatively or intrinsically distinguishable from all other such objects, and at the same time only extrinsically or quantitatively distinguishable from, and related to, each other. Unfortunately Frege debars himself from any such simple alternative account of the matter, by his deliberate refusal to consider the relevant history of the number concept, which certainly strongly supports this suggestion. And in holding instead that number is 'predicable of' concepts—*e.g.*, presumably of the concept 'leaf', among others—he is simply trying to assert in unsuitable and not merely awkward language, what he has forbidden himself to assert in suitable terms, with the result that both concepts and numbers come to be hypostatized, and the latter, at least, are thus miraculously transported across the borderline separating logic from the special sciences.

In other words, according to a most plausible alternative which Frege fails to consider, the scientific formulation of the concept of number involves a process of ideal selection and abstraction from the 'data of experience', whereby the notion of a whole (the number system) composed of related elements or 'units' ideally homogeneous with each other, and with the whole they constitute, is finally arrived at, and made an object of scientific—*i.e.*, mathematical—investigation on its own account. What renders this alternative especially attractive, not to say compelling, from the point of view of the logic of science, is that it alone squares perfectly with the generally accepted account of the way in which, *mutatis mutandis*, basic concepts of the other sciences have been logically formulated and developed. Sense perception provides the data, the raw material, for all of the natural sciences, and it is to essentially the same process of ideal selection and abstraction, differing however in the degree and extent to which it is carried out, that each of the several sciences owes its fundamental concepts. On this view numbers constitute the simplest, most elementary, and most abstract system of relationships pertaining to the phenomenal order. On the contrary, to accept the account of the Frege-Russell school of thought of the way in which the number concept is arrived at, is arbitrarily to cut mathematics off from its undeniably intimate connections with the other sciences, and is to attempt to link it instead with logic, in an unholy alliance which actually raises more problems than it solves.

And there is one more important point worth mentioning in this connection. From the point of view of mathematics as a science, and that means as a progressively developing body of knowledge, the very attempt to formulate a definitive definition of number is basically mistaken. No definition of any scientific concept can be more than provisional and temporary, and no scientific definition can have more than pragmatic sanction as a working instrumentality of the science at any given stage of its development.

So much by way of exposition and criticism of the first phase of Frege's endeavor to provide an absolutely impeccable basis for mathematical reasoning.

The second methodological and philosophical problem, that of establishing criteria for absolutely rigorous proofs, may next be approached. The best brief account of Frege's teaching on this subject is also to be found in the *Grundlagen.*

Practising mathematicians are usually satisfied if the truth of the propositions composing their science is subject to no serious doubt, without being troubled too much by the problem of establishing the strict interdependence of truths with each other, and without searching too closely into the question as to what constitutes logical rigor in demonstration. But precisely these questions weigh most heavily upon Frege's mind, and will give him no peace until he has come to some definitive conclusion about them.

The ideal of a strictly scientific method in mathematics Frege describes in the following terms. Recognizing the utter impossibility of proving all propositions, the only alternative that he can envisage is to reduce to a minimum the number of unproved propositions, and to recognize and formulate these in the most explicit manner possible. This will insure clear knowledge of the fundamental premises of the science. And secondly all methods of proof must be specified definitely in advance, for otherwise the question of the validity of any given or proposed proof cannot be settled definitively and by general agreement. If arithmetic is to make good its claim to be only a more highly developed logic, no transitions from one proposition to another not vouched for by acknowledged and absolutely incontestable logical laws, can be allowed.

In this connection Frege adopts the Kantian terminology of analytic and synthetic, *a priori* and *a posteriori* judgments, but construes these terms in his own, decidedly non-Kantian manner (*Grundlagen,* sec. 3, 87ff.). For one thing, Frege finds that Kant's classification is not exhaustive. For example, the distinction between analytic and synthetic is made to turn upon whether the predicate-concept is, or is not contained in the subject-concept. But how about cases in which the subject is a single object, or those in which the judgment is existential? In neither of these species of judgment, argues Frege, is there any question of a subject-concept at all.[1] Moreover, Kant seems to think of the concept as determined by its adjunct marks or properties only, but according to his critic this is only only one of the least fruitful ways of constructing a concept. Frege holds that scarcely one of the definitions formulated in the *Grundlagen* is of concepts composed in this fashion. Or consider the definiton of the continuity of a function. There is here no series of adjunct properties, but rather an organic unity of determinations. And what can be deduced from such a definition is not to be discovered beforehand, so that propositions embodying these consequences are according to Kant synthetic; yet (according to Frege) they can be demonstrated by pure logic, and are hence also analytic. Furthermore, more than one definition is often required for the proof of a propositon, so that it cannot be said that the conclusion follows from any one alone, and yet it may follow in purely logical fashion from the several definitions conjointly. Very often, indeed, says Frege, one first acquires the content of a proposition, and is then confronted with the difficult problem of finding a rigorous proof for the

[1] This criticism of course overlooks Kant's contention that concepts without percepts are empty and percepts without concepts are blind.

proposition. Hence it is essential to differentiate these two features of propositions from each other.

These considerations lead Kant's critic to the conclusion that the distinction between *a priori* and *a posteriori,* between analytic and synthetic, properly apply, not to the content, but to the means of justification of the judgment. From this point of view, "to call a proposition *a posteriori,* or analytic, is not to pass judgment upon the psychological, physiological, and physical conditions which have made it possible to form the content of the proposition in consciousness, nor about how another person has happened—perhaps in an erroneous manner—to maintain its truth (is Frege charging Kant with proceeding on these lines?), but upon the ultimate grounds of the justification for the maintenance of its truth". In this way the question is removed from the domain of psychology, and referred to the domain of mathematics. Thus it becomes a matter of finding the proof for the propositon, and of following this proof back to the basic truths upon which all that follows rests. If these basic truths turn out to be general logical laws, and definitions, then the proposition in question is analytic *a priori.* But if, on the contrary, it is not possible to carry out the proof without using truths referrible to a particular domain of knowledge, then the proposition is synthetic. And if the proof involves appeal to matters of fact it is *a posteriori.* According to Frege it can be shown that the propositions of arithmetic are in this sense analytic *a priori,* while those of geometry are synthetic. But if the criticisms advanced above have any validity, then this contention is highly questionable.

At all events, the two complementary contentions which have emerged so far from this study of Frege's work on the foundations of mathematics are: (1) that the basic concepts of arithmetic can be reduced, by the process of definition, to concepts of pure logic; and (2) that the basic propositions of arithmetic may be derived, by the process of proof, from purely logical premises. The *Grundesetze der Arithmetik* undertakes to justify this thesis by actually establishing the science in the manner proposed. As a means to this end the author devised a symbolism which almost succeeds in literally picturing the interconnection of the successive steps in his demonstrations. Unfortunately, however, this symbolism is so cumbersome and inflexible that it is well-nigh unworkable, and has generally been discarded by later students of symbolic logic. For this reason, no attempt will be made to introduce it, or the specific demonstrations clothed in it, into this account. Attention will be concentrated, instead, upon the logical foundations of the work as a whole.[2]

These foundations are, to say the least, very peculiar. There is no doubt that they are rather difficult to comprehend, and this fact may lead some students to suppose that they are unusually profound; but such is by no means the case. The difficulties spring, rather, from Frege's odd terminology, and from his use of familiar terms in unfamiliar and even very strange and ambiguous senses. Even so staunch an admirer as Russell himself (in his *Principles of Mathematics:* Appendix A) finds some of Frege's doctrines very curious, some untenable, and some the result of a simple confusion of psychology with logic. Russell nevertheless declares —whatever such a declaration can possibly mean in the light of these criticisms—that Frege's work "avoids all the usual fallacies which beset writers on logic"!

In an article entitled "Ueber Sinn und Bedeutung" in Volume 100 of the *Zeitschrift für Philosophie und philosophische Kritik* (1892, pp. 25-52), Frege first sets forth some conceptions that are all-important for a proper understanding of his later work. Taking 'signs' and 'names' to include any sort of notation for proper names, he in turn construes the *Bedeutung* of the latter as the definite objects (*Gegenstaende*) to which they apply. Thus 'evening star' and 'morning star' may be proper names for the same object, and in such circumstances their *Bedeutung,* their indication (to adopt Russell's translation), will be the same, but they will nevertheless differ in meaning (*Sinn*). For in the meaning is contained the mode or "way of being given"; while the indication of a proper name is the object it indicates. "A proper name (word, sign, combination of signs, expression) expresses its meaning, and indicates or designates its indication" (p. 31). Finally, it should be specially noted, proper names, so construed,

[2] Any reader who is interested in studying Frege's symbolic calculus in detail may be referred to Vol. I of Jørgen Jørgenson's *Treatise of Formal Logic.*

cannot apply to concepts or relations; separate consideration of their logical rôle will come later.

But now comes an application or extension of this doctrine which is little short of astounding. Frege observes that it is also permissible and necessary to speak of the meaning and indication of an entire assertive sentence. Some sentences, however, for example those poetic assertions which have as subjects proper names, such as 'Odysseus', which indicate no real object, seem *ipso facto* to have as a whole no indication but only a meaning (say, as part of a poem). And in this case the sentence has no value, when the question of its truth or falsity, as distinct from its meaning, arises. Thus it can be said that "it is the search for truth that inspires us to press on from the meaning to the indication" (p. 33). And conversely, to ask for the truth-value of a sentence is to seek its indication, and hence the indication of its proper parts. From this line of reasoning Frege concludes that he is compelled to recognize the truth-value of a sentence—*i.e.*, the fact that it may be either true or false—as its indication. And thus every assertive sentence is to be taken as a quasi-proper name, whose indication, if it have one, is either the true or the false. Of this doctrine Frege himself observes that this implicit denomination of truth-values as objects (*Gegenstaende*) may appear to be merely an arbitrary edict or verbal jugglery, from which no important results are to be anticipated. Yet he insists that in every judgment—*i.e.*, in every assertion or recogniton of the truth of a thought content—a transition has already implicitly been made from subjective ideas to objective indications. For "a truth-value can no more be part of a thought content than can the sun, which is not a meaning but an object" (p. 35). Furthermore, it is obvious on this view that all true propositions have the same indication, as do also all false propositions, however much they may differ in meaning; and this dichotomous division of propositions will turn out to be an important consideration from the point of view of logical demonstration as Frege conceives it.

Frege nevertheless also states that to say, 'the thought, that 5 is a prime number, is true', is to say no more and no other than that 5 is a prime number' (p. 34). But on this point Russell sharply (and rightly) disagrees. "There is great difficulty in avoiding psychological elements here, (he says) and it would seem that Frege has allowed them to intrude in describing judgment as the recognition of truth. Psychologically, any proposition, whether true or false, may be merely thought of, or may be actually asserted; but for this possibility, error would be impossible. But logically, true propositions only are asserted . . ." (*Prin. of Math.*, Appendix A).

It is, however, by means of the preceding doctrine that Frege develops his strange conception of functions, to which attention must next be directed. For it is in terms of this conception that he will explain what he means by a 'relation' as distinguished from objects and concepts. Readers who know their calculus are familiar with the mathematical concept of function, involving the idea of one or more independent variables, each capable of assuming all values in a given domain. Thus the quantity y is said to be a function of the independent variable (or argument)—symbolized by y = f (x)—or of the n independent variables (or arguments) $x_1, x_2, x_3, \ldots x_n$, if to every value or set of values which the independent variable or variables may assume, there corresponds a value of y. For example, the distance traveled by a moving body is a function of several independent variables, such as the time during which the motion takes place, the initial velocity, the accelerative force, or forces, and the resistance. And the dependent variable may remain constant while the independent variable varies, as a person's weight at successive intervals of time. But on pain of being badly confused otherwise, such readers had best rid their minds entirely of this ordinary mathematical conception, when attempting to understand Frege's doctrine, though Frege does use some mathematical terms to explain his meaning.

Russell quotes the two following statements from Frege as to the nature of a function: (1) If in an expression, whose content need not be propositional (*beurtheilbar*), a simple or composite sign occurs in one or more places, and we regard it as replaceable, in one or more of these places, by something else, but by the same everywhere, then we call the part of the expression which remains invariable in this process a *function,* and the replaceable part we call its argument. (2) If from a proper name we exclude a proper name which is part or the whole of the first, in some or all of the places where it occurs, but in such a way that these places remain recognizable as to be filled by one and the same arbitrary proper name (as argument positions of the first kind), I call what we thereby obtain the name of a function of the first order with one argument. Such a name, together with a proper name which fills the argument-places, forms a proper name (cf. *Grundgesetze*, p. 43).

As Frege remarks, this way of regarding functions extends the circle of functional values far beyond the range of numbers; and, as he does not make sufficiently clear, also alters entirely the meaning of the term. For example, taking 'the present President of the United States' as a proper name, 'United States' is the argument and 'the present President of' is the function, in Frege's terminology. Similarly, '$x > y$' is a first order function with two arguments; and by carrying the procedure a step further a second order function may be obtained: 'There is at least one value of x satisfying ϕx' is a function of ϕ, which is itself a function. Functions of the first order with two variables, it should now be carefully observed, *express relations*. This illustrates the way in which relations are to be construed generally in Frege's logical calculus. And finally, the value of the function '$x^2 = 4$' is either the truth-value of the true or the false; and '$2^2 = 4$' and '$3 > 2$' *indicate* the same truth-value, namely the true. In '$x^2 = 4$', the value is the true for the arguments '2' and '-2', and the false for all other arguments.

The next notion to be introduced is that of a range (*Wertverlauf*). The extension of a concept, Frege defines as the range of a function whose value for every argument is a truth-value (*Grundgesetze*, p. 8). In this sense the intensional aspect of concepts is logically prior to their extensional aspect, and hence he holds that Schroeder was badly mistaken in taking the opposite view. Two functions are said to have the same range when they have the same value for every value, say, of x; when, namely, for every value of x both are true or both are false.

The importance of this doctrine is, that if 'range' be identified—as Russell helpfully suggests—with what Russell calls a 'class', cardinal numbers can hereupon be defined in such a way as seemingly to unify logic and mathematics even more completely than was possible in terms of the earlier and simpler doctrines of the *Grundlagen*. Only, following Frege's intensional view, number will still be a 'property' of class-concepts, not, as for Russell, of classes in extension (and accordingly criticisms similar to those already expressed above also apply here). To quote Russell, "if *u* be a range, the number of *u* is the range of the concept 'range similar to *u*' ". On this 'logical' basis, Russell asserts, the *Grundgesetze* proves "various theorems in the foundations of Cardinal Arithmetic . . . with great elaboration, so great that it is often very difficult to discover the difference between successive steps in a demonstration".

To complete this account of this part of Frege's work it only remains to set forth his doctrine of logical implication, which is very simple indeed. According to Frege, the sole mode of drawing conclusions requisite for his demonstrations is a dichotomous relation which holds between *p* and *q* whenever 'either *q* is true or *p* is not true'—(false)—where *p* is not necessarily a proposition, and whatever *q* may be (*Grundesetze*, pp. 25ff.). *Whatever q may be*, 'the truth-value of *q*' indicates the true if *q* is true, and the false otherwise. Negation is indefinable, and belongs to the content of an assertion; and the joint assertion of *p* and *q* is the denial of '*p* implies not-*q*'. In the process of 'proof' all *further* definitions are merely nominal, in that they substitute a simple, brief expression for a more complicated one.

The statements just made, however, should not be taken to mean that *all* definitions are nominal. On this point Frege is very explicit, to the effect that mathematics requires definitions (like those of number, given above) which will enable one to decide unambiguously whether for example a given empirical thing 'falls under' a given concept or not; *i.e.*, whether the concept can be predicated unambiguously of the thing or not. And a similar test must be applied in defining relations. A plain implication of this doctrine, namely that concepts can never be subjects of categorical judgments, Frege also always maintains. Ultimate simples, of course, are indefinable, and their nature can only be made clear by some method of 'pointing out'; so that in the end, it would seem, curiously enough, all non-verbal definitions must rest upon a quasi-empirical test of givenness.

Furthermore, in addition to the one law of reasoning formulated above, Frege either implicitly or explicitly recognizes the universal validity of the usual so-called laws of thought—the laws of identity, non-contradiction, and excluded middle. Whether these laws are in some sense ultimate, or derivative from such as are—and if the latter, what these ultimate laws are—is never brought out. But it hardly seems worth while, in the light of subsequent developments in symbolic logic, to pursue a criticism of what is manifestly, in these very important respects, an unfinished piece of work.

But is it not astonishing, to anyone at all familiar with the actual intricacies of even the most rigorous mathematical reasoning, and the richly diverse procedures deemed essential to their work by the greatest mathematicians, to be told that the sole method of demonstration which ideally *ought* to be recognized is that which Frege specifies? How perverse in principle is the demand that human reason, even when occupied with the high abstractions of pure mathematics, deliberately restrict itself beforehand to any one such quasi-mechanical routine! No more than definitions fixed once and for all, would this principle of inference, if so it may be called, really serve the very complex logical requirements of a genuine science. Even the qualifications suggested by Russell in his discussion of Frege's doctrines are not such as to render the principle more adequately representative of modes of reasoning which both mathematicians and logicians recognize as both indispensable and perfectly legitimate for the purposes of even the most rigorous science. Under these circumstances, one is forced to the hard alternative of either accepting Frege's dictation and of throwing out, as a consequence, much of what would otherwise pass as perfectly valid reasoning, or of simply refusing to abide by the severe strictures which Frege would impose. And there can be little doubt as to which alternative will actually prevail in practice.

In the second volume of the *Grundgesetze*, moreover, the author gives much space to a sound criticism of formalist theories of mathematics, which hardly seems quite consistent with all that has gone before. The formalist is one who thinks of his science as closely analogous to a game such as chess, the counters or figures meaning nothing in themselves, but serving merely as tags to which arbitrarily fixed rules may be applied. According to the critic, this view forgets the difference between a theory of the game, and the game itself. "The moves of the game take place, it is true, according to the rules; but these rules are not objects in the game, but constitute the basis of the theory of the game. In other words, while the moves in chess take place according to rules, no position of the chess figures, and no move, expresses a rule; for the status of the figures in a chess game is in general not to express anything, but simply to be moved in accordance with the rules". So in *formal* arithmetic 'a + b = b + a' would have to be regarded, not as an object of the game, but as one of the rules forming the basis of the theory of the game (*Grundgesetze,* II, 114). The actual rules of arithmetic, however, are really based upon the import of the symbols, and this import, according to Frege, is none other than the content of the science. But could you not, indeed must you not go further, and by broad analogy construe logic as theory of reasoning, and mathematics (as well as every other science) as operating *with* certain material, *according* to the laws of logic? Must one not here also, much as in the case of the game, carefully distinguish between laws according to which one reasons, and premises from which one reasons—a basic distinction which it would seem that Frege had implicitly failed to observe in his own case?

At all events Frege insists that mathematics is a genuine science whose aim is truth, and not a mere game played according to arbitrary rules with meaningless counters. One potent consideration that is of itself decisive here, is the fact to which Frege calls attention, that mathematics can be applied to all sorts of problems in many fields, whereas a mere game, by its very nature, can be applied to nothing.

Such, then, in brief, are the results attained by a logician and mathematician in struggling with problems which have since been wrestled with by a host of other students of those subjects. Richly suggestive and stimulating though Frege's pioneering endeavors undoubtedly are, it must be admitted that he succeeded rather in raising such problems than in settling any of them in a tolerably satisfactory manner. Not only is his work essentially incomplete, in spite of its considerable voluminousness, but in failing to consider other possible solutions more consonant with the actual developments in mathematics as a going concern, it must be admitted that he seriously misdirected and misguided other thinkers, and that from a philosophical point of view in particular his equipment for the tasks he set himself was obviously far short of being adequate. Neither in formulating his definitions of numbers, functions, and the like, nor in his attempted identification of (objective) truth with truth-values, nor in his theory and practice of mathematical reasoning, does he reveal profound appreciation of the philosophical issues involved, much less succeed in meeting those issues squarely.

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