Ockham's Razor and Plato's Beard (Or, "The Possible Relevance of the Philosophy of Mathematics, and the Problem of Universals in Particular, to the Philosophy of Mathematics Education, and the Problem of Constructivism in Particular")
Author(s): William of Ockham and Robert E. Orton
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OCKHAM'S RAZOR[1] AND PLATO'S BEARD[2]

(*Or*, "The possible relevance of the philosophy of mathematics, and the problem of universals in particular, to the philosophy of mathematics education, and the problem of constructivism in particular")

WILLIAM OF OCKHAM[3], *Lake Wobehere State University*
with the assistance of
ROBERT E. ORTON, *University of Minnesota*

William of Ockham responds from the dead to an article appearing in the January 1992 issue of the *Journal for Research in Mathematics Education*, in which Paul Cobb, Erna Yackel, and Terry Wood propose a "constructivist alternative to the representational view of mind." Ockham, now a convert to Platonism, argues three points. First, that by opposing construction to representation, Cobb et al. misinterpret the postepistemological perspective of Richard Rorty's 1979 influential book, *Philosophy and the Mirror of Nature*. Second, that by opposing mathematics in the students' mind to mathematics in the environment, and, in particular, by attempting to argue that the representational theory of mind opens the "learning paradox," Cobb et al. misinterpret Carl Bereiter (1985), confuse ontological and epistemological issues, stumble into the perennial philosophical problem of universals, and indicate that they might be interested in discussing the philosophy of mathematics. Third, that in arguing for a relatively pure, "radical" constructivism, Cobb et al. mistake the pragmatic force of the constructivist argument, confuse matters of value with matters of taste, and attempt to fashion too dogmatic a connection between theory and practice in mathematics education.

It is hard for me to stay dead, though by now you would think it is something I should be good at, having had over 600 years of practice. However, Purgatory is a mean between extremes (Life and Death), and having never been fond of needless distinctions, I find it once again necessary to try to say something to this World. What has awoken me from my slumber this time is an article appearing in the January 1992 issue of the *Journal for Research in Mathematics Education*, in which Paul Cobb, Erna Yackel, and Terry Wood propose a constructivist alternative to the representational view of mind. As I admired the work of Cobb et al. (1992), my earthly vanity was once again aroused, seeing in their arguments a striking resemblance to the problems discussed by myself and my colleagues (Thomas Aquinas, Duns Scotus,

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[1]Dictum of theoretical parsimony, for example, "plurality is not to be posited without necessity" (*pluralites non est ponenda sine necessitate*), used by Ockham and subsequent thinkers to keep entities and distinction to a minimum. Compare C. S. Peirce: "[T]here is no distinction of meaning so fine as to consist in anything but a possible difference in practice" (Peirce, 1878/1965–66).

[2]"Nonbeing must in some sense be, otherwise what is it that there is not? This tangled doctrine might be nicknamed *Plato's beard*; historically it has proved tough, frequently dulling the edge of Ockham's razor" (Quine, 1953, pp. 1–2).

[3]*Venerabilis Inceptor*. English philosopher born c. 1280, venerated for his insights in psychology, metaphysics, and logic. Accused of heresy in 1323, Ockham fled to Bavaria in 1328 and was excommunicated in 1331. He remained a student (*inceptor*) rather than a doctor of theology until his death in 1349.

and Peter Abelard, to name a few) earlier in your millennium. I will be arguing below that many of the ideas in Cobb et al. are not new. These ideas have a very long intellectual lineage and can be best understood against the background of the historical tapestry into which my name is forever woven.

Accordingly, I have enlisted the help of a mathematics educator with an interest in philosophy, who will assist me in making my ideas more public. In what follows, I will argue that the recent paper by Cobb et al. is caught in that state of Purgatory with which I am all too familiar. On the one hand, Cobb et al. want to take the philosophy of mathematics education into the 21st century and beyond with their radical constructivism. But on the other hand, their attempt is still caught up with the problem of universal terms of the 13th century. I will not be proposing a solution to Cobb et al.'s dilemma. Indeed, there is nothing that would please me more than to have them stay with me here in Purgatory for a few thousand years, debating issues near and dear to me. Nonetheless, in the final section of this paper, I will venture to offer some advice.

In Part I, I will speak to Cobb et al. from the future (or here, postmodern) perspective. My main point will be that Cobb et al., in arguing against the representational view of mind, enlist the help of a philosopher whom they would better leave alone, assuming that they are in the business of "proposing alternatives" (as opposed to doing therapy). This is the philosopher Richard Rorty, who is still very much alive and with whom I thus have not (as yet) had the opportunity to speak personally. Nonetheless, I have studied his influential book, *Philosophy and the Mirror of Nature* (1979), in which he appears to be arguing against a representational view of mind but is really fighting a much grander battle. Namely, Rorty is arguing against the entire discipline of epistemology, especially Kant's constructivist alternative to the Cartesian model of mind.

In Part II, I will speak to Cobb et al. from the past, arguing that there is much in common between their proposed alternative to a representational view of the mind and 13th century problems of universals, as well as 20th century problems in the philosophy of mathematics. Specifically, I will argue that Cobb et al. are caught in Purgatory because they are stuck on ontological and epistemological issues that my colleagues and I debated over 600 years ago. This is nowhere more evident in their paper than where they discuss the "learning paradox," a truly sticky problem in the field of universals. I will show that their argument—that the learning paradox results when one adopts a representational view of mind and disappears when one adopts a constructivist alternative—rests on a confusion between questions about being and questions about knowledge. I won't venture to "solve" the learning paradox. Both Plato (c. 370 B.C./1956b) and Bereiter (1985) have attempted to do so. Instead, I will argue that the issues raised by the paradox are more appropriately discussed within the context of 13th century problems of universals or 20th century philosophy of mathematics, not as a quibble between construction and representation.

Finally, in Part III, I will explain why Cobb et al. are trapped in Purgatory. Without giving away the argument completely, Cobb et al. are trapped because they want to construct a theory that is consonant with their values—a sensible idea indeed,

if one believes in my Razor. However, in this particular instance, I believe that Cobb et al. would do better to distinguish theoretical from practical issues and to temper the urge to fashion an "epistemology" consistent with their practice, because the risk in trying to do so is dogmatism in educational theory or obfuscation in educational practice. Accordingly, in this last Part, I will be arguing for a tepid, or "wishy-washy" form of constructivism, which might also be viewed as insipid, impudent, insolent, or critical, depending on one's taste.

PART I

Radical Constructivism and the Postepistemological Perspective

> Constructivism is, logically, a postepistemological perspective. The standard questions of epistemology cannot be answered—or even reasonably asked—from this perspective. Its premises suggest, rather, abandonment of traditional epistemological language. (Noddings, 1990, p. 18)

In their recent article in the *Journal for Research in Mathematics Education*, Cobb et al. (1992) argue that there is tension between a constructivist and a representational view of mind. Citing Ernest (1991), they note:

> The term constructivism itself covers a panoply of theoretical positions. Some of these appear to be eclectic positions in which researchers attempt to combine the notion of learning as active construction with aspects of the representational view of mind. (Cobb et al., 1992, p. 3)

The implication here is that there is something wrong, from a theoretical point of view at least, with trying to be eclectic. Doing so leads to "an implicit appeal to two incommensurable semantic theories" (e.g., constructivist and objectivist semantics) (p. 6). The tension between these theories "indicates that the representational view of mind as applied to mathematics education suffers from a deep-rooted conceptual anomaly" (p. 6). "Further, the resolution of this difficulty would seem to require a major theoretical restructuring" (p. 7).

Thus, Cobb et al. imply that there is an opposition between construction and representation. They support this view using arguments from psychology, anthropology, and pedagogy.

> Psychologically, [the representational view of mind] falls prey to the learning paradox .... Anthropologically, it fails to consider the social and cultural nature of mathematical activity .... Pedagogically, it leads to recommendations that are at odds with the espoused goal of encouraging learning with understanding. (p. 2)

I will examine Cobb et al.'s pedagogical argument against the representational view of mind in Part III and their psychological/anthropological arguments in Part II. In Part I, however, I want to focus more generally on the relationship between Cobb et al.'s "constructivist alternative" and what Noddings (1990) calls postepistemology.

I must confess that in many ways I am ill-qualified to speak about postepistemology, because Purgatory runs, like your Earth, on a time frame. The future is not revealed

to us. Furthermore, I am deeply suspicious of all this postmodern, poststructuralism, postepistemology nonsense. But I have read some of Rorty's work, have argued a little with Wittgenstein about whether or not there are genuine philosophical problems, and have listened to Heidegger meditate about the nature of Being. From the ways that these philosophers eschew "foundations," I find it puzzling that Cobb et al. would attempt to enlist Rorty to support an *alternative* to the representational view of mind, because "supporting alternatives" is what "being epistemological" (as opposed to "being postepistemological") is all about. This point will be expanded below.

As I understand Rorty, however, there are no constructivist alternatives to the representational view of mind. Yet there are alternatives to "being epistemological." What are they? Again, I must confess that I can't make sense of any of them—the ones that I have looked at end up making language or culture or history or something else "foundational," but you have to remember where I am coming from—I am not the best character witness on this point. On the other hand, I *fear* that there may be some alternatives to being epistemological. Though the future is hidden from us as from you, I have noticed that our Director has a special liking for Noddings's work, and thus I tend to believe that Noddings is well qualified to speak about eschatological matters. The following section is an attempt to gloss Noddings's remark that constructivism is a "post-epistemology" (Noddings, 1990, p. 18).

Rorty and Postepistemology

In his influential book *Philosophy and the Mirror of Nature*, Rorty (1979) compares the present situation regarding a "theory of knowledge" (be it constructivist, rationalist, or empiricist) to the situation that Enlightenment philosophers found themselves in with regard to the problems discussed by medieval philosophers. The endless medieval discussions regarding the status of universal terms and related arguments debated by myself and my colleagues (sometimes parodied as questions about how many angels could dance on a pin) simply did not interest Enlightenment philosophers. This was not because Enlightenment philosophers—continuing the parody—had finally succeeded in counting the number of angels on a pin. Nor was it because Enlightenment philosophers were proposing alternative solutions to the scholastic problems. They simply had no interest in universals, angels, or pins. They wanted to change the subject.

Rorty says essentially the same thing about epistemological squabbles today (such as those that Cobb et al. are engaged in between representational views of the mind and constructivist alternatives). Whether knowledge is essentially a construction, as Kant argued, or whether it is formed by the process of Nature writing her story on Locke's *tabula rasa*—this, according to Rorty, is no longer an interesting problem. Trying to decide between representational and constructivist alternatives to the mind is similar to trying to determine whether 5 or 500 angels can dance on a pin.

Rorty makes the argument even stronger. The point is not just that epistemology is no longer interesting. Epistemology exists as a discipline of inquiry only because philosophers are confused about the place of philosophy in culture. Rorty is

opposed to the claim that philosophers can say anything constructive or systematic (as opposed to therapeutic or edifying). Thus, his target is not a particular philosophical theory but a theory about philosophy—"philosophy as a discipline [that] sees itself as the attempt to underwrite or debunk claims to knowledge made by science, morality, art, or religion" [and one could add mathematics education] (Rorty, 1979, p. 3).

As opposed to this foundationalist (constructive or systematic) view of philosophy, Rorty describes the view implicit in the work of his "heroes" (Dewey, Wittgenstein, and Heidegger):

> Each tried, in his early years, to find a way of making philosophy "foundational".... Each of the three came to see his earlier effort as self-deceptive, as an attempt to retain a certain conception of philosophy after the notions needed to flesh out that conception (the 17th century notions of knowledge and mind) had been discarded. Each of the three, in his later work, broke free of the Kantian conception of philosophy as foundational, and spent his time warning us against those very temptations to which he himself had once succumbed. Thus their later work is therapeutic rather than constructive, edifying rather than systematic, designed to make the reader question his or her own motives for philosophizing rather than to supply him or her with a new philosophical program. (Rorty, 1979, pp. 5–6)

Thus, philosophy cannot propose "alternatives," in Rorty's view. It can, at best, help one engage in self-therapy. As Wittgenstein puts it: "What is your aim in philosophy?—To show the fly the way out of the fly-bottle" (Wittgenstein, 1958, section 309).

The "fly-bottle," within the present context, is based on what Rorty takes to be a mistake common to the work of philosophers from Locke to Descartes to Kant to the present day. The mistake is a belief that one can find something interesting to say about the status of a knowledge claim (a question of justification) by looking at the origins of belief (a question of genesis). This "confusion" (according to Rorty) reaches its zenith when Kant mixes up predication (an issue for logicians) with synthesis (an issue for psychologists) (Rorty, 1979, pp. 148–155). But questions about how human beings justify their knowledge claims are different from questions about how the mind works. It is a mistake, in Rorty's view, to try to gain insight into the validity of beliefs by proposing a theory of mind ("poking around in internal representations"), even if this theory of mind be a "constructivist alternative."

Thus, Rorty's target is much grander than a representational view of mind. His target is a Kantian conception of philosophy. In this regard, Rorty's review of the work of another postmodern philosopher, Derrida, applies as much to Rorty as to Derrida himself:

> [Derrida] admits that all this stuff about there not being any such thing as accuracy of representation is *metaphorical*, just a way of speaking.... His attitude towards centuries of worry about the relation between subject and object, representations and the real, is like the Enlightenment attitude toward centuries of worry about the relation between God and man, faith and reason.... Derrida is trying to do for our highbrow culture what secularist intellectuals in the 19th century tried to do for theirs. *He is suggesting how things might look if we did not have Kantian philosophy built into the fabric of our intellectual life, as his predecessors suggested how things might look if we did not have religion built into the fabric of our moral life.* (Rorty, 1982, pp. 97–98, italics in original)

In other words, Rorty's target is not the representational view of mind but the very idea of a philosophically interesting theory of the mind. That he mentions "representation" so often is an historical accident. (It just happens to be the view most cogently supported by current philosophers of mind.) "Representation" is, in many ways, merely collateral damage.

My point here is that Cobb et al.'s opposition between a "representational view of the mind" and a "constructivist alternative," at least insofar as this opposition is fueled by Rortian postepistemology, is a mistake. Rorty is of absolutely no help in proposing constructivist alternatives (though he is fond of deconstructive literary analysis). He does not oppose "representation" to "something else" (e.g., a "constructivist alternative"). Rorty opposes "philosophically interesting theories of mind" to the work of poets, novelists, and playwrights who also write, though not in a systematic way, about the nature of human cognition. For Rorty, the real battle lies in an undiscovered country, from whose bourn no epistemological traveler has yet returned (and retained an interest in epistemology, anyway). In this never-never land, "theory of knowledge" as a constructive discipline has withered away and been replaced with the notion of "philosophy as therapy." If Cobb et al. really want to follow Rorty, they will forever forswear using the "e" word—epistemology.

A Different Perspective: Hilary Putnam

My intent above is to suggest that Cobb et al. have got Rorty wrong. My intent is not to criticize Rorty. Still I must confess that I find Rorty's program disquieting. Rorty does not believe that there are interesting alternatives to the metaphor of mind as a "mirror of nature." Yet it is hard for me not to believe in alternatives. To argue that intellectual discussions are never constructive but, at best, therapeutic tends to make the game less fun to play. For one who is not ready to go postepistemological and can only see through the glass darkly, Rorty is taking away the fun from a child.

I think it is fair to say that *not* all present-day philosophers, and not most dead philosophers, are postepistemological. Among the dead, to be sure, there are some exceptions (certainly Nietzsche, maybe Hegel). Among the living, however, many are deeply sympathetic to Kant's constructivism, yes, to Kant's "representational view of mind." My assistant needs to brush up on the *Critique of Pure Reason* (1787/1929) before we can talk more about the way in which Kant artfully blends construction and representation. While he is doing so (it may take some time), let me mention the work of one living thinker whom Cobb et al. attempt to enlist in their argument against the representational view of mind—a philosopher who, I believe, shares Kant's view (and who is opposed to Rorty's view) that there are still interesting conceptual questions related to the nature of mind. This is the philosopher Hilary Putnam.

Cobb et al. imply that Putnam and Rorty are both opposed to representation in the philosophy of mind. This is at best a half-truth. Putnam and Rorty are fighting different battles. Their hearts do not beat to the same drummer. For example, Putnam reacts to Rorty's suggestion that epistemology is dead as follows:

> Philosophers are beginning to talk about the great issues again, and to feel that something can be said about them, even if there are no grand or ultimate solutions. There is

> an excitement in the air. And if I react to Professor Rorty's book [*Philosophy and the Mirror of Nature*] with a certain sharpness, it is because one more "deflationary" book, one more book telling us that the issues aren't deep and the whole enterprise was a mistake, is just what we don't need right now. (Putnam, 1983, p. 236; cited also in Gardner, 1987, p. 77)

Putnam would probably welcome Cobb et al.'s attempt to say something constructive about epistemology. But Putnam's book, *Representation and Reality* (1991), which Cobb et al. cite to support their "constructivist alternative," is not really an attack on the representational view of mind. Putnam is arguing against the *functional* view of mind. Though there are some interesting parallels between the functional theory of mind and what Cobb et al. call the "representational view of mind," the views are *not* identical. For example, many functionalists would not support the strong dualism between mind and world that Cobb et al. want to attribute to "representationalists" (cf., Churchland, 1986, p. 37).

What is particularly relevant for the present discussion, however, is that Putnam was (about 20 years ago) perhaps the leading *proponent* of the functional theory of mind. Now he says that he was wrong. That is, Putnam's target in *Representation and Reality* is a former manifestation of Putnam. He writes:

> I am ... criticizing a view that I myself earlier advanced. Strangely enough, there are philosophers who criticize me for doing this. The fact that I change my mind in philosophy has been viewed as a character defect. When I am lighthearted, I retort that it might be that I change my mind so often because I make mistakes, and that other philosophers don't change their minds because they simply never make mistakes. (Putnam, 1991, p. xii)

In the same vein, Putnam cites his teacher, the philosopher Rudolf Carnap: "'I used to think ... I now think' was a sentence construction that was ever on Carnap's lips" (Putnam, 1979, p. xii).

In Part III, below, I will explore in greater detail the difference between a theoretical position that must be constantly subject to revision and a values position that must be the subject of commitment. As a sort of advance organizer, I want to recommend Putnam's (and Carnap's) *tentativeness* as a model for a "wishy-washy" constructivism. In Part II, however, I want to discuss the cognitive/epistemological issues raised by Cobb et al. in greater detail. My goal will be to lure them deeper and deeper into Wittgenstein's fly-bottle, that is, into scholastic discussions regarding the nature of knowledge and the foundations of reality.

PART II

The View From The Past: Why Plato Was Right

> [A] universal is not a substance existing outside the mind ... a universal is a mental content .... Just as a spoken universal is a genus or a species, but only by convention, in the same way the concept thus mentally fashioned and abstracted from singular things previously known is universal by its nature. (Ockham, c. 1320/1964, pp. 38–46)

Yes, yes, I know I said that. Please don't remind me. Even (or maybe especially)

a philosopher is entitled to a few mistakes. Besides, the errors of great people are more significant than the truths of the not-as-great, and though I am now convinced that a universal is a substance existing outside the mind, I hope my error of over 600 years ago is pardonable. Or at least my error should be understandable, because it is essentially the same error made by the radical constructivist mathematics educators of your late 20th century when they argue that mathematics is a human construction. Well, like all of us, their earthly days are numbered, and soon they will see that there are more things in Heaven and Earth than are dreamt of in the philosophy of the nominalists and the conceptualists.

For the truth is, Plato was (or I should say *is*) right.[4] He was a little off when it came to the proportions of the Divided Line (Plato, c. 370 B.C./1956b, 508D–511E). (The Good is even more extensive than he imagined, a point that I am painfully reminded of every moment.) However, his basic analogy pretty well describes the spiritual realm where my colleagues and I now find ourselves. In other words, just as shadows and other reflections are images of real earthly objects, such as tigers and trees, so mathematical ideas and other human mental constructs are images of the true ideas or universals that exist in this supersensible realm (Plato, 508E–510A).

Put another way, learning mathematics *is* "a process in which students modify their internal mental representations to construct mathematical relationships or structures that mirror those embodied in external instructional representations" (Cobb et al., 1992, p. 2). To this I would only add that these external representations are themselves copies of even more perfect representations that exist in a supersensible reality. In this ontological realm, for example, you will either find those four integers, x, y, z, and n, all greater than 2, such that $x^n + y^n = z^n$, or you will not find these integers. Don't expect me to tell! What is more, during my 600 years in Purgatory I have discovered (with the assistance of Pierre de Fermat) a remarkable demonstration pertinent to this question.[5] Unfortunately, the space of this article, not to mention the intellect of my assistant, is too small to accommodate it.

In a minute, I will say more about this supersensible reality, but let me first indicate the overall direction of this second part of the paper. In what follows, I will argue two points that essentially contradict each other. My first point will be that Cobb et al. are probably confused when they describe their "constructivist alternative to the representational view of mind," this confusion being most apparent in their mixing up epistemological and ontological issues when discussing the "learning paradox." My second point will be that Cobb et al. might not be confused after all—that there might be some reason for mixing up what is *known* and what *is*—but that doing so opens the possible relevance of the philosophy of mathematics, and the problem of universals in particular, to the philosophy of mathematics education, and the problem of constructivism in particular.

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[4]Of course, I shouldn't expect anyone who is still alive to believe me. The impressions received by inmates of Purgatory are very different from those you are now experiencing, and it is *only* from this view from beyond that I can now see, all too clearly, that there is a Reality.

[5]Rumor has it that your English mathematician, Andrew Wiles, has also now discovered a demonstration. Pierre is checking on Wiles's proof as we speak.

I prefer to accept the first point and will spend most of Part II arguing in support of it. The argument will turn on a tidy distinction between epistemological and ontological inquiry—a distinction that is also congenial to von Glasersfeld's (1985) position that constructivists would do well to avoid ontological commitments. But the second point is also interesting. If argued more fully than can be done in this paper, I believe it would show that Cobb et al.'s "constructivist alternative" has something in common with the constructivism of Brouwer and the mathematical intuitionists, not to mention my celebrated work on universal terms.

The Problem of Universals

This section is irrelevant if Cobb et al. really are confused. But just in case they do intend to plead clarity and argue that there is no interesting distinction between *knowing* and *being*, let me briefly introduce some ontological skeletons in the closet.

Purgatory is a treacherous and crowded place, compounded by the existence of these sharp, three-sided figures that we inmates keep bumping into. (Polygons with more than three sides are not as hazardous.) Not only will you find here the Form of Tigerness, which each individual tiger who lives or who has ever lived necessarily participates in. You will also find the Form of Triangleness, which is responsible for the existence of each individual triangle you draw and each perfect triangle that you can imagine. This leads to the (human) question of how the individual tiger or triangle is related to its respective universal Form.

Let me re-express this question. The word *triangle* is used, today as it was two millennia ago, to refer to any one of these three-sided, closed figures that can be drawn on paper, pictured in the mind, or (recently) illuminated on computer screens. But how can one word be used to pick out an infinity of individuals? Undoubtedly, the individuals or particulars all share something in common, or at least resemble each other in some way. But what is the nature of this common thing or resemblance?

This is one of several philosophical problems associated with so-called universal terms. A universal term is any noun (in, say, the English language), with the exception of a proper name. The term is "universal" because it can be used to refer to any number of particular instances. For example, the term *tiger* can be used to describe any number of large, striped, sharp-toothed meat eaters that live in China, India, or the city zoo. The term *triangle* can be used to denote any number of the infinitude of "ideal triangles" that we spiritual beings keep bumping into.

The different human perspectives on universal terms are complex, but historians of philosophy often summarize these positions as three—realism, nominalism, and conceptualism.[6] The difference between these "isms" hinges on the intuitive distinctions between *world*, *language*, and *mind*. Roughly speaking, proponents of each of the three positions on the problem of universals locate the "common thing" in one of these three domains.

[6]If Cobb et al. are not confused, then I would argue that they are buying into conceptualism.

The first position, which is attributed to Plato, is that universal terms are essentially part of the furniture of the world (not necessarily *your* world). Thus, this position is called realism. Though no one can see a universal tiger or a universal triangle, these entities exist as indubitably, though perhaps not in the same way, as particular tigers or particular triangles. Plato argued that these universals exist in the World of Forms. Augustine followed Plato's lead. However, many of my medieval colleagues who were influenced by Aristotle backed off from the World of Forms somewhat, arguing that universals exist in things (*universalia in rebus*), not independent of particulars (*universalia ante rem*). In other words, all triangles contain the real "essence of triangleness." If this distinction between the particular and the essence of the universal seems convoluted to you, so it seemed to me, and I excised it with my celebrated Razor.

A second position, which was evident in ancient times but is often associated with my name, places the universal not in the world but in language. According to this position, a universal is a linguistic convention or name. Universal terms like *tiger* and *triangle* are just convenient names that human beings use to organize their experience and make sense of the world. Hence this position is usually called nominalism. Proponents of nominalism might appear to put the universal in the mind, and indeed this is where I ultimately place it as well. However, strict nominalists, in a manner similar to that of your logical behaviorists and mathematical formalists, do not think that postulating "mind" serves any explanatory purpose, at least with respect to the problem of universals. Specifically, the operations of the mind are essentially identical to the operations of language. Or, as one of my English colleagues, Thomas Hobbes, put it sometime after me: "*Ratio est oratio* (thought is speech)" (Hobbes, 1651/1968, part 1, chap. 4).

Proponents of a third position on universal terms attempt to negotiate a truce between the nominalists and the realists. They argue, roughly, that a universal term is more than a linguistic convention but less than an independently existing entity or essence. Specifically, the universal term *triangle* refers to an idea in the mind of the term's user. Thus, the position is usually called conceptualism. The origins of conceptualism can actually be traced to my work. For example, I argued:

> [T]he intellect apprehending a singular thing performs within itself a cognition of this singular only. This cognition is called a state of mind, and it is capable of standing for this singular thing by its very nature. (Ockham, 1320/1964, p. 47)

But three of my English colleagues (John Locke, George Berkeley, and David Hume) are usually given credit for conceptualism. Locke writes:

> The mind makes the particular ideas, received from particular objects, to become general .... [T]hus universals, whether ideas or terms are made. (Locke, 1706/1961, book II, chap. 11, section 9)

I do not want to get into a proprietary battle at this point over who invented conceptualism. What is interesting, for the present discussion, is that my erroneous position is similar in many or most respects to that of the radical constructivists in mathematics education. Specifically, we share the same error of attempting to

overcome the environment-mind dualism by placing mathematics in the students' activity. Thus I argued, anticipating Piaget by six centuries: "A concept is the same as the act of knowing ... the act of intellect, by its very nature, and without any convention, stands for the thing to which it refers" (Ockham, 1320/1964, pp. 38–46). If only I had had a glimpse of Plato's World of Forms, as Augustine apparently did, when I was alive!

The Learning Paradox

There are at least two basic problems with universal terms. One problem has just been discussed and is summarized by questions such as the following: What is the nature of the common thing shared by, say, five triangles or seventeen tigers? Where does this "common thing" exist? Does it exist in this world or in some World of Forms (realism), in language (nominalism), or in human thought (conceptualism)? These are essentially ontological questions, and I will return to them near the end of Part II.

There is also another set of problems associated with universal terms that arise in a more direct way in Cobb et al.'s paper. These questions are concerned not with how the common thing *exists* but how the common thing becomes *known*: How do we come to know the "common thing" shared by five triangles or seventeen tigers? Do we apprehend universals by recollecting them (Plato), by abstracting them from particulars (myself, 600 years ago, as well as Locke, Berkeley, and Hume), or by constructing them (Kant, Piaget, and, as I understand it, Cobb et al.)? These are not ontological but epistemological questions.

This second set of questions includes one that is particularly sticky. Namely, how can we have knowledge of a universal by means of particulars without knowing the universal to begin with? For example, to grasp a concept such as *tiger* or *triangle*, it is necessary to have some knowledge about individual tigers or triangles. But to recognize that individual tigers or individual triangles all share something in common, it is necessary to understand the respective concept to begin with. This is the so-called learning paradox. Cobb et al. formulate it as the "problem of explaining how students make constructions compatible with those that the expert has in mind" (Cobb et al., 1992, p. 9). Considering the relationship between work with manipulatives (Dienes blocks) and paper-and-pencil algorithms, they write:

> The assumption that students will inevitably construct the correct, internal representation from the materials presented implies that learning is triggered by the mathematical relationships they are to construct before they have constructed them.... How then, if students can only make sense of the world in terms of their internal representations, is it possible for them to recognize mathematical relations that are developmentally more advanced than their current internal representations? (Cobb et al., 1992, p. 5)

This paradox has a very long history. It was lucidly stated in perhaps its earliest form by Plato, in his dialogue *The Meno* (Plato, c. 370 B.C./1956a), in which Menon and Socrates discuss the problem of how a concept (specifically, the concept of virtue) might "trigger" its learning before it has been learned. As usual, Socrates pretends ignorance, saying in effect that the concept of virtue is "developmentally more advanced than [his] current internal representation." To this Menon responds:

> And how will you try to find out something, Socrates, when you have no notion at all what it is? Will you lay out before us a thing you don't know, and then try to find it? Or, if at best you meet it by chance, how will you know this is that which you did not know? (Plato, 79D)

To which Socrates replies:

> I understand what you wish to say, Menon. You look on this as a piece of chop-logic, don't you see, as if a person cannot try to find out either what she knows or what she does not know. Of course she would never try to find what she knows, because she knows it, and in that case she needs no trying to find; or what she does not know, because she does not know what she will try to find. (Plato, 79D)

Thus, for Plato, the learning paradox arises from the fact that, in order to learn something new, one must already know something old. This leads to the question as to how the something old is related to the something new. If the two are identical, then one "would never try to find what she knows, because she knows it." If the two are different, then "one does not know what she will try to find," and thus one cannot learn the new thing.

Plato "solves" the learning paradox in a way that sounds fanciful to modern ears. Namely, he argues that human learners are able to grasp a structure more complex than they presently know because they have, buried within their minds, remembrances of past lives which have been washed not-quite clean in the River of Neglectfulness (Plato, c. 370 B.C./1956b, 612E–620E). In other words, we are able to grasp structures more complex than we currently know because we begin to remember our past lives. As one commentator describes Plato's solution:

> The human soul has prenatal knowledge of universals and of their mutual relations, and postnatal experience of the ordinary world serves, or may serve, to revive this knowledge in suitable circumstances. Thus, experience does not directly provide us with new apprehensions (of universals) or with new knowledge of necessary truths (connections between universals) but acts as a stimulus to remind us of what we already know but have hitherto in this life forgotten. (Woozley, 1967, p. 198)

This theory of recovery, as fantastic as it appears, is embraced in a modified form by serious theorists today, such as Chomsky and Fodor.

One of the most cogent reviews of the learning paradox as it applies to present-day educational thought was done by Bereiter (1985). In what follows, Bereiter's formulation of the learning paradox, and Cobb et al.'s reading of Bereiter, is examined in greater detail.

Cobb et al. and Bereiter

Bereiter begins his analysis of the learning paradox in tones that would likely warm the constructivists' souls, but then he quickly turns heuristically critical.

> A core belief in contemporary approaches to learning is that knowledge and cognitive strategies are actively constructed by the learner (Wittrock, 1974). This belief came to prominence in educational thought through the influence of Piaget (Ripple & Rockcastle, 1964) .... Although this "constructivist" view of learning and development has a great deal of intuitive appeal and empirical support, it is in trouble theoretically. It seems

> to be generally agreed that there is no adequate cognitive theory of learning—that is, no adequate theory to explain how new organizations of concepts and how new and more complex cognitive procedures are acquired (e.g., Lockhart, 1982; Newell, 1980; Norman, 1980; Voss, 1978). Sensitive to this problem, Piaget (1977a, 1977b, 1978) spent the last years of his career attempting to specify the processes by which knowledge takes place in the child. Nevertheless, in his celebrated debate with Chomsky (Piatelli-Palmarini, 1980), Piaget and his collaborators were flatly accused of having no theory to show how cognitive growth is possible. Although Piaget argued forcefully for the *necessity* of a constructivist account of development and intelligence, it seems to be acknowledged by most of the scholars involved in the debate that no convincing theoretical answer had been provided. (Bereiter, 1985, p. 201)

Thus, as Bereiter suggests, the learning paradox is a challenge for constructivists. He quotes Fodor's (1980) succinct statement of the challenge a few lines later:

> There literally isn't such a thing as the notion of learning a conceptual system richer than the one that one already has; we simply have no idea of what it would be like to get from a conceptually impoverished to a conceptually richer system by anything like a process of learning. (Fodor, 1980, p. 149)

Bereiter's approach to this challenge is positive and hopeful. Specifically, he offers six strategies or "heuristics" for tackling the learning paradox (p. 206). These include investigating the mechanisms behind conceptual "bootstrapping," avoiding "premature reductionism," breaking with the mind-as-computer metaphor, assuming a "rich innate endowment," investigating "chancy attainment" of complex cognitive structures, and assuming that "teachability implies bootstrapping" (pp. 206–207).

Bereiter also considers one possible "solution" to the learning paradox that plays an important role in Cobb et al.'s paper. This approach emphasizes the social or cultural aspects of mathematical knowledge growth. As Cobb et al. put it: "The [learning] paradox seems less ominous once we broaden our focus and acknowledge the naturalness of certain interpretations is a product of our own acculturation into the mathematical practices of society" (pp. 9–10). Cobb et al. would appear to be following Vygotsky's lead here:

> More generally, it is by capitalizing on students' mathematical activity that the teacher initiates and guides the classroom community's development of taken-as-shared ways of mathematical knowing that are compatible with those of the wider community. The paradox that Bereiter raises then disappears. It is only a paradox if one separates knowledge from the knower and considers that learning is a process of apprehending mathematical relationships that are self-evident only to the initiated. (Cobb et al., p. 11)

Bereiter's development of this anthropological point, however, is less optimistic:

> It is tempting to try to circumvent the learning paradox by shifting the analysis of learning and development to a cultural level. Following Vygotsky (1978), one might formulate the following explanation: Learning does, indeed, depend on the prior existence of more complex cognitive structures, but these more complex cognitive structures are situated in culture, not in the child. The child acquires them through interaction with adults, who help the child do things that it could not do alone. Through such shared activities the child internalizes the cognitive structures necessary to carry on independently. (Bereiter, 1985, p. 206)

Yet, Bereiter (and Vygotsky himself) deny that this social process of "appropriation" solves the learning paradox. Bereiter continues:

> Such an explanation [e.g., that appeals to enculturation or "appropriation" to solve the learning paradox], satisfying as it may appear, does not eliminate the learning paradox at all. The whole paradox hides in the word "internalizes." How does internalization take place? It is evident from Luria's first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved before there could be an adequate social-cognitive theory. Solving that problem means confronting, not circumventing, the learning paradox. (Bereiter, 1985, p. 206)

In other words, social constructivism does not solve the problem. In effect, Cobb et al. misunderstand the "punch" of the learning paradox. They attribute this problem, described by Plato more than 2000 years ago, debated by Piaget and Chomsky in this century, and studied by Piaget in the last few years of his life, to a dualism between mathematics in the environment and mathematics in the student's mind. They attempt to "solve" the learning paradox by appealing to sociocultural mechanisms of knowledge acquisition that even Vygotsky and his group acknowledged needed more work. One might suggest, charitably, that Cobb et al. have found the key to the learning paradox that eluded both Vygotsky and Piaget. But charity has never been my strong suit (another reason why I am here), and I prefer to believe that Cobb et al. are confused.

Of course, it is not very helpful to suggest that someone is confused without showing just *how* he or she is confused. In the next section I will argue that Cobb et al.'s confusion turns on a supposed "opposition" between constructivism and realism.

Constructivism and Realism

According to Cobb et al., it is the separation between "internal representations ... located in students' heads and external representations located in the environment ... that underlies Bereiter's (1985) formulation of the learning paradox" (Cobb et al., 1992, p. 7). Bereiter, on the other hand, puts the problem as follows: "How can the development of complex mental structures be accounted for by mechanisms that are not themselves highly intelligent or richly endowed with knowledge?" (Bereiter, 1985, p. 205). I want to argue that it is the separation between "less complex" and "more complex," between "less intelligent" and "highly intelligent," between "less endowed" and "richly endowed," that underlies Bereiter's (and Plato's) formulation of the learning paradox. What is at stake is the *complexity* of the representation that is being learned. The *location* of the representation is another matter.

In Cobb et al.'s formulation of the learning paradox, however, "complexity of representation" and "location of representation" are conflated. The paradox arises not just because the child's cognitive network is "structuring" from a less to a more complex state, during which time the "plan" for this change is not present in the old state. The paradox also arises because the plan is allegedly located (according to proponents of the "representational view of mind") in the manipulative materials that are external to the mind of the learner. "Because concepts more complex than those that the learner has available are *located in a preformed environment*, it is unclear how the learner should grasp them" (Cobb et al., 1992, p. 7, italics added).

Thus, I want to distinguish two issues:

1. The problem of explaining how a change from a less complex cognitive state to a more complex cognitive state is possible
2. The problem of accounting for the location of the "learning plan" that describes the more complex cognitive state

It is true that these two problems might be related. However, complexity of representation need not be related to location of representation. The first problem raises an issue of *knowing*, whereas the second is an issue of *being*. Or, to use the typical philosophical jargon, the first problem is epistemological, whereas the second is ontological.

I don't believe that Cobb et al. really want to become engaged in an ontological discussion. Constructivists "deliberately and consequentially avoid saying anything about ontology, let alone making any ontological commitments" (von Glasersfeld, 1985, p. 100, cited in Kilpatrick, 1987, p. 10). My problem is that I can't reconcile von Glasersfeld's assertion with Cobb et al.'s treatment of the learning paradox. It seems to me that if you don't want to become engaged in an ontological discussion, then you don't want to argue that your dialectical opponent is getting into trouble (e.g., falling victim to the learning paradox) because of an ontological distinction. Following Rorty, you want to change the subject.

The problem is that Cobb et al. and other radical constructivists want to set up an opposition between constructivism and realism. Von Glasersfeld writes:

> Throughout the two thousand five hundred years of Western epistemology, the accepted view has been a realist view. According to it, the human knower can attain some knowledge of a really existing world and can use this knowledge to modify it. (von Glasersfeld, 1990, p. 27)

I want to argue, on the other hand, that there is no necessary opposition between constructivism and any ontological position. If "realism" is regarded as a metaphysical position, as it is in standard philosophical discussion, then realism is opposed to both nominalism and conceptualism (in the 13th century), as well as idealism (in the 18th and 19th centuries). The radical constructivists' opposition between realism and constructivism (which takes the form of an opposition between a "representational view of mind" and a "constructivist alternative" in Cobb et al.'s paper) shows, at best, a lack of historical sense. At worst, it shows a state of conceptual confusion.

My point is thus really the same as von Glasersfeld's. Constructivists "deliberately and consequentially avoid saying anything about ontology, let alone making any ontological commitments" (von Glasersfeld, 1985, p. 100, cited in Kilpatrick, 1987, p. 10). From this it follows, to my logic, that constructivism is consistent with *either* "metaphysical realism" or idealism. And since constructivism is consistent with metaphysical realism, it cannot be opposed to it.

The "Possible Relevance" Argument

On the other hand, if both von Glasersfeld and I are arguing the same point (e.g., that constructivism is not an ontological position), and yet I believe in metaphysical

realism and he eschews it, then something seems a little strange. Can I be a radical constructivist and a metaphysical realist?

Earlier I voiced my support for a realist or a Platonist view of mathematics. Proponents of this position locate mathematical reality in some "external domain." This is the view that Cobb et al. parody as "immaculate perception"—that mathematical meaning is somehow transparently inherent in the world (Cobb et al., 1992, p. 3). My support for the realist position, though to be taken with a grain of salt, is a serious attempt to try to make sense of Cobb et al.'s position. In other words, my goal is to try to understand their "constructivist alternative" by comparing it to a position that appears to be different from theirs—the realist view of universal terms.

If the realist view of universal terms *is* relevant for describing what constructivism is not—in other words, if Cobb et al. *do* want to become engaged in an ontological discussion—then I want to argue that the philosophy of mathematics, and the problem of universals in particular, may be relevant to the philosophy of mathematics education, and the problem of constructivism in particular. Let me expand on this "possible relevance" idea, following a train of thought sketched by W. V. O. Quine. Quine (1953) suggests that the ontological questions debated by my colleagues and me surfaced, once again, during the earlier parts of your 20th century in the foundations of mathematics. According to Quine, the question of whether universals are located in nature, in language, or in the mind corresponds to the three positions often used to organize the philosophy of mathematics. These are the positions of realism, represented in the work of Frege, Russell, and Whitehead, among others; formalism, represented in the work of Hilbert; and intuitionism, represented in the work of Poincaré, Brouwer, and Weyl, among others.

It is perhaps no accident that the conceptualist-intuitionist position is often described as "constructivist" (cf., Davis & Hersh, 1981). Do constructivists such as Brouwer and constructivists such as von Glasersfeld, Steffe, and Cobb share more than a name? Quine's description of conceptualism-intuitionism would appear to resonate in many ways to that of the radical constructivists.

> Conceptualism holds that there are universals but they are mind-made. Intuitionism, espoused in modern times in one form or another by Poincaré, Brouwer, Weyl, and others, countenances the use of bound variables to refer to abstract entities only when those entities are capable of being cooked up individually from ingredients specified in advanced. As Frankel has put it, logicism holds that classes are discovered while intuitionism holds that they are invented—a fair statement indeed of the old opposition between realism and conceptualism. (Quine, 1953, p. 14)

Though I would not want to saddle the radical constructivists in mathematics education with intuitionism in the philosophy of mathematics, I believe that they are taking a conceptualist position on universal terms. Whether this conceptualism extends to a position about the philosophy of mathematics is the question.

My point is this. If Cobb et al. do want to push the contrast between constructivism and metaphysical realism, then they will engage the ontological problem of universal terms, described earlier in Part II. Cobb et al. might respond that they want nothing to do with this issue. But if this is the case, then I would argue that they will need to drop the claim that constructivism is opposed to "metaphysical realism." This move,

as far as I can tell, would also do away with their realist "foil"—the representational view of mind.

Summary and Conclusion

Realists argue that universal terms are part of the furniture of the universe. Nominalists argue that universals merely play a part in the language game of a culture (much as Wittgenstein argues that the meaning of a term is the way in which the term is used). Conceptualists argue that universals are learned by constructing them in the active mind. The relevance of this classical tapestry, for Cobb et al.'s argument, is this: The ontological twist they put on the learning paradox makes it a question of how humans come to know universal terms. Specifically, in order to grasp a universal concept, it is necessary to have knowledge of particulars. But, in order to grasp that things are particulars of the universal, one must already have the universal concept.

Unlike Wittgenstein, I do not believe that this problem with universal terms turns on a mere pseudoproblem—a problem of language. I believe that it is a significant philosophical problem, inherent in the conceptual landscape, much like a geological fault or an interesting "singularity" in complex analysis or topology. But is the existence of this singularity (i.e., the problem of universal terms) *relevant* to constructivism in mathematics education? Or, to express the question in the form of the refrain that captures much of the spirit of this paper: Is the philosophy of mathematics, and the problem of universals in particular, relevant to the philosophy of mathematics education, and the problem of constructivism in particular?

I believe that the answer to this last question is "no." But, once again, the only way I can make sense of Cobb et al.'s opposition between a "representational view of mind" and a "constructivist alternative" is to see them as engaging this classical issue (viz., the problem of universal terms). By confounding, in their formulation of the learning paradox, the problem of conceptual growth with the problem of the nature of conceptual reality, Cobb et al. either accidentally or intentionally stray into ontological waters. I would be delighted if they want to stay there. But I do not think this is where they want to be.

To repeat: The learning paradox arises irrespective of one's ontological commitments. It is not a consequence of a "representational view of mind." It also plagues any "constructivist alternative" that one might choose to formulate. If one embraces a realist ontology, then the learning paradox might be "solved" in several different ways. One might argue, as Plato did, that the "plan" for the learning is inherent in the mind from a previous life. One might argue, as Chomsky does, that the plan is somehow innate. Or one might argue, as Fodor does, that the plan is somehow part of the "language of thought." On the other hand, if one embraces a conceptualist ontology, one might argue, as I did 600 hundred years ago, that the plan arises in the process of thought. Or one might argue, as Kant did, that the plan arises in the synthetic unity of apperception. Finally, one might argue, as Piaget attempted to do in the last few years of his life, that the plan arises from interactions between previous and new cognitive structures.

Why do Cobb et al. conflate epistemological issues of learning mathematics with ontological issues about the nature of mathematical entities? Barring confusion, I believe the reason is that Cobb et al. are not ready to go postepistemological. Though they imply that they want to follow Rorty and his proponents, they are still caught up in the problem (as am I, let me emphasize) of justifying knowledge claims. They imply that they want to have nothing to do with "foundations." Yet they are still in the business of "proposing alternatives" and then trying to justify these alternatives by conceptual arguments. They claim to eschew "permanent, neutral frameworks," but, by mixing ontological and epistemological issues, I believe that they are trying to find one nonetheless.

You can't be postepistemological and at the same time worry about epistemological problems, such as the learning paradox. Either the learning paradox, when used as an argument against the representational view of mind, is a veritable red herring (the postepistemological perspective), or raising the paradox indicates a desire to engage in a certain type of philosophical discussion (the "possible-relevance" perspective). In the former case, Cobb et al.'s psychological argument against the representational view of mind collapses, and their support for a "constructivist alternative" turns on an instructional or pedagogical point. Here, I have no quarrel whatsoever. I will have more to say about this in Part III. In the latter case, however (i.e., assuming the learning paradox is *not* a red herring), Cobb et al.'s argument shows me that they are interested in fishing in ontological waters. If this is true, then their arguments are best located within the context of an historical debate on the status of universal terms and the nature of mathematical entities.

PART III

The Values Issue
(Or, "Tepid Constructivism: Why Some Like it Wishy-Washy"[7])

> Truth is disputable, not taste: what exists in the nature of things is the standard of our judgment; what each person feels within him or herself is the standard of sentiment. Propositions in geometry may be proved, systems in physics may be controverted, but the harmony of verse, the tenderness of passion, the brilliancy of wit must give immediate pleasure. (Hume, 1751/1957, slightly modernized, p. 5)

Cobb et al. might be stuck in Purgatory, somewhere between Rorty's postepistemology and Ockham's Razor. However, my intent is not to offer them a way out of Purgatory. I am delighted with their company. Of course, the feeling might not be mutual—they may not share my sentiment or taste—and if this is the case, then I have tried to show in Parts I and II that they might move in at least one of two directions. Either they could move in the direction of Rorty, which would involve dropping the entire epistemological discussion, or they could move in the direction of more traditional, academic philosophy, which would involve calling up all

[7]c.f., von Glasersfeld (1990), "An Exposition of Constructivism: Why Some Like it Radical."

sorts of ghosts and goblins. Though I prefer the latter alternative, I realize that this is merely a matter of taste. This paper is wishy-washy in that it only attempts to point out the problem. I do not want to suggest a solution.

My advice to Cobb et al. is that they can go in the direction of Rorty and postepistemology if they wish, but they should not try to support their move in that direction by showing that they are avoiding a traditional epistemological puzzle, specifically, the learning paradox. Rorty is postepistemological, not because he is trying to "solve" a traditional paradox. Indeed, as I read his book, he no longer finds epistemological problems of interest. No. If the learning paradox and the dualism between mathematics in the environment and mathematics in students' heads are their real interests, then they will need to debate the issues in more traditional terms. This will involve a sober examination of some familiar philosophical terrain, such as the relationship between problems of knowledge (epistemology) and problems of being (ontology), the status of universal terms, and the philosophy of mathematical entities. This alternative requires that they call up the ghost of Kant. For, as Rorty argues, no one can be taken seriously as a philosopher unless he or she has mastered the first *Critique* (Rorty, 1979, p. 149).

Parts I and II have focused primarily on the epistemological/cognitive aspect of constructivism. I have said virtually nothing, up to this point, about methodological/pedagogical issues. My reason for silence regarding these latter issues is that I have absolutely no quibble whatsoever with Cobb et al.'s values position. No mathematics educator who wishes to be taken seriously would. Their positions on cognitive science and epistemology, on the other hand, are another matter.

In this last section I frame my difference with Cobb et al. as a matter of taste, picking up on Hume's idea that truth, and not taste, is disputable. In the process, I also attempt to understand Cobb et al.'s pedagogical argument against the representational view of mind (viz., that it "leads to recommendations that are at odds with the espoused goal of encouraging learning with understanding" [Cobb et al., 1992, p. 3]). Specifically, I argue two things. First, that Cobb et al.'s argument against the representational view of mind turns not on a conceptual point, but on a pragmatic one. Their argument, when taste is stripped from substance, comes down to the pragmatic point that it is better for mathematics educators to believe in a constructivist view of mind than a representational alternative. My second point, related to the first, is this: Though it might be better for mathematics educators to *believe* in a constructivist model of the mind, this belief is ill-supported by epistemic or ontological arguments, for these latter turn on taste, not truth.

Values and Tastes

The tastes of a spirit, dead for 600 years, should obviously be taken with a grain of salt. But I hope that salt is not the only taste that remains in the reader's mouth, at this point. Here, it is necessary to speak frankly about one possible source of misunderstanding regarding the wishy-washy constructivism that I am supporting. Matters of taste are different from matters of truth. But matters of value are different from both. One can be critical of constructivism in mathematics education and still

embrace wholeheartedly the values of mathematics educators throughout the world. These values, which every mathematics educator knows in his or her soul, pertain to the importance of children's learning mathematics in a meaningful manner. That this is a values perspective, as opposed to a factual matter, is obvious from only a fleeting conversation with many research-oriented educational psychologists or measurement specialists in state departments.

Thus, I would like to draw a working distinction between "values" and "tastes" (i.e., "tastes" as "positions on theoretical issues"). Issues of taste are personal, idiosyncratic, likely dependent upon personal history and upbringing, and, except for close associations between people (e.g., marriage), likely to be matters of individual freedom or even personal whim. Issues of value, though they share many features with issues of taste, are concerned primarily with matters of conviction, commitment, promise, steadfastness, and character. Issues of value may turn on some sort of cause, crusade, program, or "higher calling."

On these matters of value, there seems to be little disagreement within the mathematics education community. Talk of "community," in fact, would be nonsense without fundamental agreement about values. Mathematics educators basically agree with the intent of the *Standards*, even though there may be some skirmishes about details. But these are minor issues. Following Brownell (1935), mathematics educators oppose rote to meaningful learning, purposeless to purposeful activity, and a classroom driven by management concerns to one in which learning mathematics is of primary importance. Mathematics educators seem to be universally opposed to the drill-and-practice, skill-based conception that was given perhaps its firmest theoretical support by Thorndike (1922). In other words, there appears to be pretty good community agreement as to what "meaningful" is (or at least what it isn't).

Mathematics educators may agree, universally, that children should learn mathematics in a meaningful manner. But this agreement does not derive from the fact that mathematics educators are all convinced on some scientific point. On the other hand, the agreement is more than a matter of taste. It is a matter of commitment to the education of children, which is an issue of values.

Just because one questions the way in which Cobb et al. attempt to dress up in philosophical garb a relatively sensible idea of learning—namely, that the learner must be actively involved in the process (an idea that was espoused at the inception of mathematics education as a discipline in the middle 1800s)—it doesn't follow that one has a different values orientation. Just because one ventures to question the way in which Cobb et al. attempt to anchor a relatively sensible idea about the importance of "learning with understanding" with lofty theories from metaphysics and epistemology, it doesn't mean that one is a "behaviorist," with all the nasty connotations that mathematics educators attach to the term. Just because one questions the attempt to find a noneclectic, relatively pure, "radical" exposition of constructivism, it doesn't follow that one favors a return to a lockstep curriculum in which each student is forced to recite the times tables until he or she achieves 95% accuracy.

No. Differences in theoretical positions do not necessarily translate into differences in values, and vice versa. The former is a matter of taste, whereas the latter is a matter of commitment.

The working hypothesis here is that there is no necessary connection between the realm of theory and the realm of values. This is not a hypothesis that there is no connection, only that, whatever type of connection there is, it is not a logical or necessary one. From this it follows (logically) that one can criticize constructivism and argue that "some like it wishy-washy" without disagreeing with the basic, sensible idea that children need to be actively involved in the learning process. What is questioned here is the broad, philosophical gloss that is placed over this rather tepid point about learning.

My point is that there is no "right" philosophical gloss to the conviction, shared by mathematics educators, that children need to be actively engaged in their own learning. The position of the radical constructivists may be, in many ways, the best thing for mathematics educators to believe. But this is a *pragmatic* point. It is very different from a *conceptual* move, that is, a philosophical argument that children really construct their own knowledge. The next section will explore this pragmatic point in greater detail.

The Limits of Pragmatism

I believe that it is better for me to *believe* that knowledge is constructed by the learner, irrespective of whether this epistemological outlook is true or false. If Cobb et al.'s argument for their "constructivist alternative" stopped at this pragmatic point, I would have no further rejoinder. But they appear to want to make more of this values issue than is possible. Rumpelstiltskin notwithstanding, one cannot necessarily spin theoretical gold from value-laden hay.

By raising questions of transcending Cartesian dualisms, the learning paradox, and the representational view of mind, Cobb et al. indicate to me that they are trying to mount more than just a pragmatic argument for their "constructivist alternative." Specifically, it seems to me that they are trying to support constructivism with some sort of conceptual argument, based on philosophical notions of realism, learning, and ontology. It is here that I begin to get uneasy—actually a bit dizzy—as the historical ground moves underfoot and the discussion shifts from values to tastes.

Where I think Cobb et al. and the radical constructivists go astray is not in the fact that they adopt a conceptualist view of universals. "'Tis not solely in poetry and music, we must follow our taste and sentiment, but likewise in philosophy" (Hume 1739/1992, book. I, part iii, section viii). What is more, when I was alive my ontological tastes were similar, if not identical, to theirs. I was fed up with Scotus's distinction between formal and nonformal identity, with Aristotle's notion that essences are simultaneously part of and not part of an individual, and with various other ontological and epistemological matters. My preference was for the Razor: to cut away needless distinctions and to focus on the fact that the world is composed of (or constructed from) particulars. When I became an inmate of Purgatory, however, my tastes changed, largely because I came across many of those same distinctions,

now embodied in a real, albeit incorporeal, manner, which now my Razor cannot slice. My tastes changed. They had to change.

Though I now see all too clearly that conceptualism is false and Platonism right, I can't expect living readers to be struck by the same impressions received by us inmates of Purgatory. I can certainly understand how Cobb et al. and the radical constructivists are falling into the same trap that I did 600 years ago. It is true that I am a little irked that the radical constructivists are rediscovering my conceptualist account of universal terms without giving me credit. On the other hand, I am delighted that they, too, are falling into this fly-bottle, not only because imitation is the sincerest form of flattery, but also because I am deeply sympathetic to the intent behind their "constructivist alternative" to a representational view of mind.

But being sympathetic to the *intent* behind Cobb et al.'s constructivist alternative is not the same as being swayed by their conceptual *arguments* for this position. And I find their arguments less than convincing.

With Cobb et al. and the other radical constructivists, I, too, believe that children construct their own mathematical knowledge. But my belief has nothing to do with "representational alternatives," Cartesian dualisms, and learning paradoxes. My belief is wholly pragmatic. It has to do simply with the fact that it is better for me to believe that children construct their own mathematical knowledge than to believe the opposite.

Let me express this last point in more stochastic terms with the help of Pascal's celebrated wager. Pascal (1662/1966, #418) showed that it is far more rational to bet on God's existence than to bet against it, given the infinite fire-and-brimstone pain of a false negative as compared with the finite "blue skying" stupidity of a false positive. Similarly, I believe it is far more *rational* to bet that children construct their own knowledge than to bet against it. If I lose when I bet that children do construct their own knowledge, then all I lose is maybe a bit too much time spent trying to care for the mathematical understanding of the child. This is a small price for a mathematics educator to pay. On the other hand, if I lose when I bet against the constructivist idea that children construct their own knowledge, I am guilty of an overly authoritarian infringement on the child's right to learn. This is a damnable offense for a mathematics educator.

Essentially the same point was made by William James, who suggests that we are justified in believing things if it is better for us to believe them:

> Often enough our faith beforehand in an uncertified result *is the only thing that makes the result come true*. Suppose, for instance, that you are climbing a mountain, and have worked yourself into a position from which the only escape is by a terrible leap. Have faith that you can successfully make it, and your feet are nerved to its accomplishment. But mistrust yourself, and think of all the sweet things you have heard the scientist say of *maybes*, and you will hesitate so long that, at last, all unstrung and trembling, and launching yourself in a moment of despair, you roll into the abyss. In such a case (and it belongs to an enormous class), the part of wisdom as well as of courage is to *believe what is in the line of your needs*, for only by such belief is the need fulfilled. (James, 1896/1956, p. 59, italics in original)

Applied to the particular case at hand, the belief that children construct their own mathematical knowledge is more likely to lead me to create fruitful learning environments

for children. The belief that children are receptacles of time-honored truth, on the other hand, might lead me to create a situation where children fail.

I take the main point of Cobb et al.'s article to be not their *epistemological* claim that knowledge is a construction, but their *pedagogical* belief that children and adults both learn best when they are actively engaged in the process. Though I share their pedagogical belief, which is underscored by experience helping children and adults learn mathematics, I am deeply skeptical of their epistemological arguments, as I tried to show in Part II. As a very tentative, wishy-washy piece of advice, I want to suggest that Cobb et al. and the radical constructivists would be better off if they would stick to the pragmatic argument, distinguish theoretical from practical issues, and not get caught up in the need to fashion an "epistemology" that is consistent with their practice. Though it is certainly true that good practice is often guided by good theory, it is exceedingly difficult to argue from the former to the latter. Furthermore, constructivism need not be supported by philosophical argument. It is "supported" just fine by pragmatic belief.

Conclusion: Why I Like It Wishy-Washy

In this paper I have attempted to illustrate the tension in Cobb et al.'s "constructivist alternative to the representational view of mind." Viewing from the postmodern perspective, I have argued that Cobb et al. misunderstand the direction in which they attempt to move. Postmodernism, as Rorty (1979) describes it, has nothing to do whatsoever with alternative theories of mind. Postmodernism attempts to describe a new relationship between philosophy and culture, one in which there is no need for a "theory of mind," be it constructivist or a representational alternative. In effect, by trying to describe a "constructivist alternative," Cobb et al. remain caught in the traditional epistemological problems that Rorty argues are no longer interesting.

Viewed from the perspective of the past, on the other hand, Cobb et al.'s attempt to frame a constructivist alternative to a representational view of mind makes much more sense. The history of epistemology (not postepistemology) can be looked upon as a series of alternatives to different positions, and Cobb et al. do appear to be taking on an epistemological position. My belief is that they are also taking on an ontological position. Specifically, they are warming over the age-old question of the status of universal terms, adopting a version of my conceptualist position with all its attendant problems.[8]

As it stands, thus, Cobb et al. are stuck with me in Purgatory. Right now, they are somewhere between Rorty's postepistemology and Ockham's Razor. On the one hand, they might move in the direction of Rorty, which would involve dropping the entire epistemological discussion (and perhaps taking up the writing of poetry or novels). On the other hand, they might move in the direction of more traditional, academic philosophy, which would involve confronting the ghosts of thinkers past.

My "position," with respect to our dilemma, is unabashedly wishy-washy. My preference is to engage the dusty issues raised by one direction of Cobb et al.'s paper

[8]Plato was right, in more ways than one. Learning appears to be a process of (historical) recollection.

while, at the same time, indicating that it might be "interesting" to see where Rorty leads, as well. But I have no taste for the ionosphere, at least not for extended periods of time. Right now, I do not choose to follow Rorty into the who-knows-where undiscovered conceptual never-never land where epistemological problems become, at best, quaint curiosity pieces, much like the medieval discussions of how many angels could dance on the head of a pin.

Postepistemology is a fascinating theoretical topic, but, like the age-old problem of universals, it is also a topic that must be approached with conceptual arguments. My wishy-washiness (i.e., tentative, noncommital position) is appropriate for engaging in speculative arguments for or against a position regarding postepistemology or a theory of universals. However, it is exceedingly difficult to translate into firm values for practice—in this case, the practice of teaching children mathematics. My only *non*-wishy-washy piece of advice here is that mathematics educators can get along just fine without philosophical theory. In fact, they probably can get along better without it.

The agreement on values among the mathematics education community should not lead to a push for a monolithic, dogmatic, or noneclectic theoretical foundation to support this agreement. In more ecclesiastical institutions, perhaps, the dangers of heresy are great enough to necessitate dogmatic prescriptions and cautions against tepid or lukewarm theoretical foundations. I erred here while alive and am now paying dearly for it. But being wishy-washy about one's values is different from being wishy-washy about one's theoretical commitments. Nietzsche said this perhaps better than anyone when he argued that consistency in a thinker is more a symptom of a disease than a theoretical virtue.

Conceptual (or philosophical) positions are a matter of taste, and there is nothing wrong with changing tastes. But this is not the same as changing values. My fling with metaphysical realism is a matter of taste. It is not a matter of commitment. Besides, word has it that, ontologically, there is even *less* in Heaven than is dreamed about by us inmates of Purgatory. (No need for universals—then we will know *even* as we are known.) I'll believe it when I see it. Or rather, when I don't see it.

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AUTHOR

ROBERT E. ORTON, Peik Hall, 159 Pillsbury Drive S.E., University of Minnesota, Minneapolis, MN 55455; rorton@maroon.tc.umn.edu